Grothendieck-Riemann-Roch & Floer homotopy

(joint w/ Noah Porcelli)

§1. Background/motivation

$X \rightsquigarrow$ graded Liouville manifold (i.e. $2c_1(X)=0$).

Symplectic cohomology of X,

$$SH^*(X;R) \equiv H_*(SC^*(X;R)),$$

is traditionally an R-module w/ R a discrete ring e.g. $\mathbb{Z}$ or $\mathbb{C}$; recall that the codifferential is defined via counting 0-dim moduli spaces of inhomogeneous pseudoholomorphic cylinders asymptotic to 1-periodic Hamiltonian orbits.

Floer homotopy aims to lift SC^* to more exotic ring spectra, e.g. sphere $\mathbb{S}$ or complex cobordism MU, by utilizing coherent tangential structure on higher-dim moduli.

in exact setting, in "general" derived MUP needed

Most generally got MU-module $\mathbb{F}^{MU}$ — this is because moduli admit coherent stable a.c.s.

If X admits stable $\mathbb{R}$-polarization, i.e., $TX \oplus \underline{\mathbb{C}}^d \cong \Lambda \otimes_{\mathbb{R}} \mathbb{C}$, get spectrum $\mathbb{F}^{\mathbb{S},\Lambda}$ — this is because can use Λ to get coherent stable framings (which depend on Λ).

Rmk:

$$SC^{-*}(X;\mathbb{Z}) \cong \mathbb{F}^{MU} \otimes_{MU} \mathbb{Z}.$$

Analogously,

$$SC^{-*}(X;\eta) \cong \mathbb{F}^{\mathbb{S},\Lambda} \otimes_{\mathbb{S}} \mathbb{Z},$$

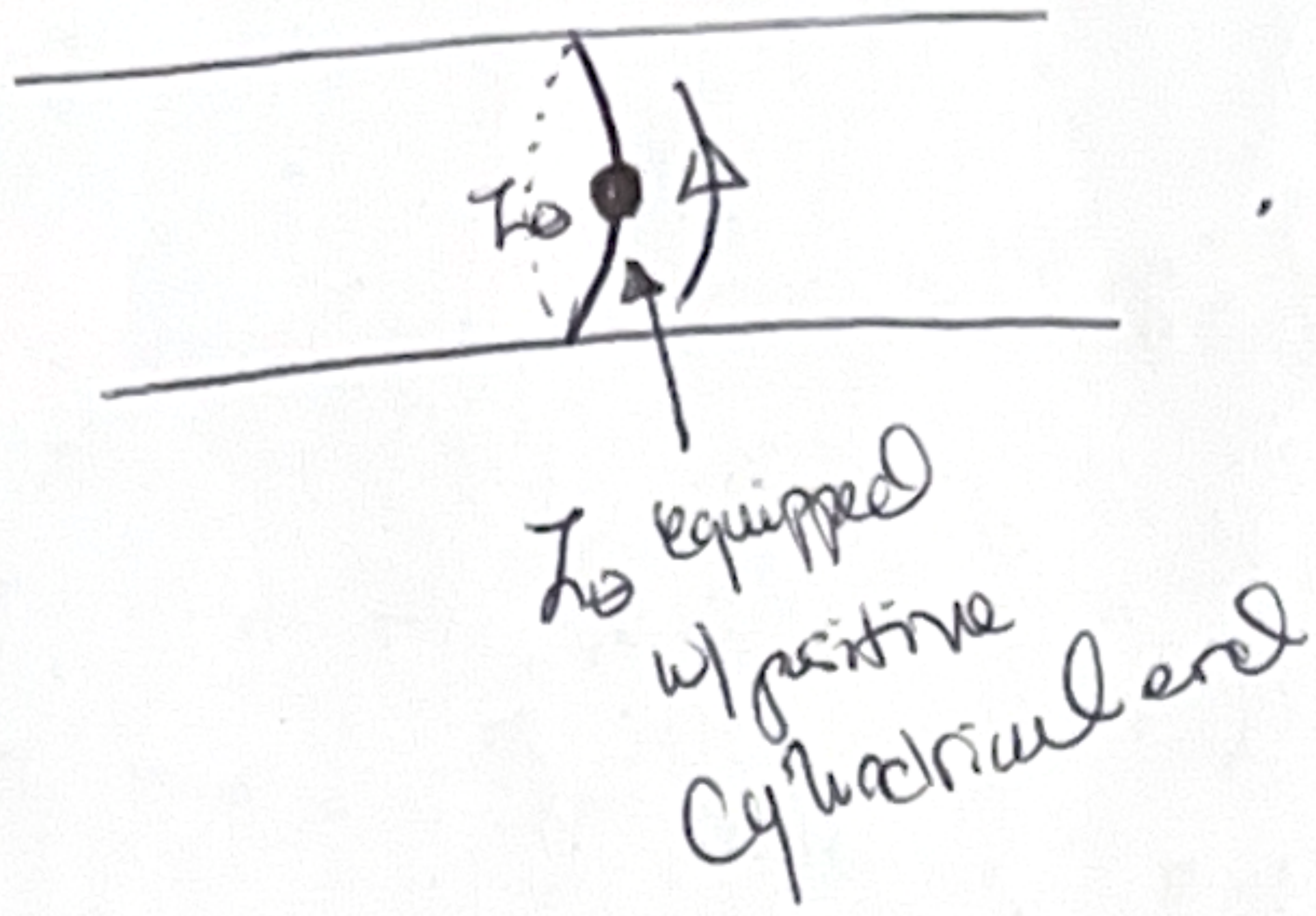
where η is a local system depending on $w_1(\Lambda)$ & $w_2(\Lambda)$.

Here is a natural question.

Q: When is $\mathbb{F}^{MU} \simeq \mathcal{X} \otimes_{\mathbb{S}} MU$ for some spectrum $\mathcal{X}$? In particular, when is $\mathbb{F}^{MU} \simeq \mathbb{F}^{\mathbb{S},\Lambda} \otimes_{\mathbb{S}} MU$ (when the RHS is defined)?

§2. obstruction to $\mathbb{S}$-lift

Recall, SH^* admits a Lie bracket $[\cdot, \cdot]$; this can be defined directly by considering a parametrized moduli space



Let

$$Ch(X) = \sum_{p \geq 0} Ch_p(X) \in H^*(X; \mathbb{C})$$

be the Chern character of X w/ $Ch_p(X)$ the $2p^{th}$ graded piece.

Thm I | (B.-Porcelli)

Suppose $\exists p_0$ s.t.

$$[\cdot, Ch_{p_0}(X)] : SH^*(X; \mathbb{C}) \to SH^*(X; \mathbb{C})$$

is non-zero, then $\mathbb{F}^{MU} \not\simeq \mathfrak{X} \otimes_{\mathbb{S}} MU$ for any spectrum $\mathfrak{X}$.

Example:

$T^*\mathbb{C}P^n$, $n \geq 3$ & odd. This follows by Viterbo + String topology computation. cf. Hepworth '09.

We will return to this example later.

§ 3. ($\mathbb{C}$-linearized) MU-Floer homotopy

Theorem I is proven by investigating

$$\pi_* \mathbb{F}^{MU} \otimes_{\mathbb{Z}} \mathbb{C};$$

this is a $\mathbb{C}[\underline{b}]$-module w/

$$\mathbb{C}[\underline{b}] \equiv \mathbb{C}[b_1, \dots, b_p, \dots] \cong \pi_* MU \otimes_{\mathbb{Z}} \mathbb{C}, \quad |b_p| = 2p.$$

Porcelli & I built a chain-level model for this, as follows.

$H^\ell \in C^\infty(X)$ ⇝ non-degenerate, linear @ $+\infty$ Hamiltonian

$\mathcal{X}(H^\ell)$ ⇝ set of 1-periodic Hamiltonian orbits

$\mathbb{F}_{yx}$ ⇝ moduli space of Floer trajectories connecting $x \in \mathcal{X}(H^\ell)$ to $y \in \mathcal{X}(H^\ell$

Again, $\{\mathbb{F}_{yx}\}$ coherently, stably a.cx. Let $ch(\mathbb{F}_{yx}; \nabla_{yx}) \in \Omega^*(\mathbb{F}_{yx}; \mathbb{C})$ represent $ch(\mathbb{F}_{yx})$, say, via Chern-Weil theory w.r.t. Hermitian connection ∇_{yx}. Consider

$$ch(\mathbb{F}_{yx}; \mathbb{C}[\underline{b}]) \equiv \sum_{p \geq 0} ch_p(\mathbb{F}_{yx}; \nabla_{yx}) b_p \in \Omega^0(\mathbb{F}_{yx}; \mathbb{C}[\underline{b}]).$$

By choosing $\{\nabla_{yx}\}$ appropriately $\Rightarrow$ $\{ch(\mathbb{F}_{yx}; \mathbb{C}[\underline{b}])\}$ are coherent.

Define

$$CM_*^\ell(X; ch) \equiv \bigoplus_{x \in \mathcal{X}(H^\ell)} \mathbb{C}[\underline{b}]\langle x \rangle$$

$$\partial_y \equiv \sum_{x \in \mathcal{X}(H^\ell)} \left(\int_{\mathbb{F}_{yx}} ch(\mathbb{F}_{yx}; \mathbb{C}[\underline{b}]) \right) \cdot x,$$

then

$$\pi_* \mathbb{F}^{MU} \otimes_{\mathbb{Z}} \mathbb{C} \cong \operatorname{colim}_{\ell} H_*(CM_*(X; ch)).$$

§4. Bulk-deformations & GRR

Meanwhile, base-changing $\Pi_X \mathbb{F}^{HM} \otimes_{\mathbb{Z}} \mathbb{C}$ along $\mathbb{C}[b] \to \mathbb{C}$ yields $SH^*(X;\mathbb{C}) \Rightarrow$ $\Pi_X \mathbb{F}^{HM} \otimes_{\mathbb{Z}} \mathbb{C}$ is a deformation of $SH^*(X;\mathbb{C})$.

A classical source of deformations of SH^*, relevant in mirror symmetry, is bulk-deformations. We'll consider the $\mathbb{L}$-module,

$$\mathbb{L} \equiv \mathbb{C}[b_1^\vee, \ldots, b_{n-1}^\vee],\ |b_p^\vee| = -2p \ \&\ 2n = \dim X,$$

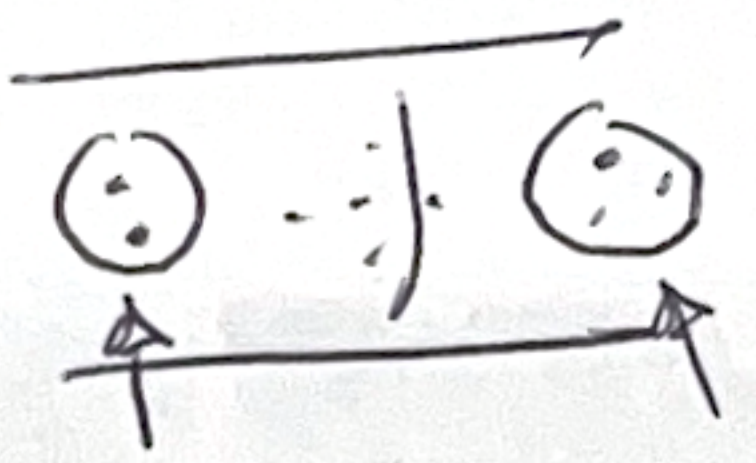
given by $SH^*(X;\mathbb{C})$ bulk-deformed by $\{Ch_p(X)\}$: $SH^*_{Ch}(X;\mathbb{L})$.

This is obtained by deforming the SC^* codifferential, as follows.

Singular model:

Consider

$$y \longmapsto \overset{\rightharpoonup}{\sum_{x \in \mathcal{X}(H^\ell)}} \frac{1}{i_1! \cdots i_{n-1}!} \# \{$$



$$\} (b_1^\vee)^{i_1} \cdots (b_{n-1}^\vee)^{i_{n-1}} x$$

i_1 marked pts hitting cycle P.D. to $Ch_2(X)$

i_{n-1} marked pts hitting cycle P.D. to $Ch_{n-1}(X)$

& take colimit over ℓ.

de Rham model:

Consider $\Pi_{yx}: \operatorname{int}\mathbb{F}_{yx} \times (\mathbb{R} \times S^1) \to \operatorname{int}\mathbb{F}_{yx}$ &

$$y \mapsto \sum_{x \in \mathcal{X}(H^\ell)} \frac{1}{i_1! \cdots i_{n-1}!} \left\{ \int_{\mathbb{F}_{yx}} \Pi_{yx,!}\, ev_{i_1}^* Ch_2(X)^{\wedge i_1} \wedge \cdots \wedge \Pi_{yx,!}\, ev_{i_{n-1}}^* Ch_n(X)^{\wedge i_{n-1}} \right\} (b_1^\vee)^{i_1} \cdots (b_{n-1}^\vee)^{i_{n-1}} x$$

evaluation @ i_1 marked pts

evaluation @ i_{n-1} marked pts

& take colimit over ℓ.

Both models equal, but de Rham is relevant for our proof.

Thm$^{\underline{n}}$ I (B.-Bonelli)

$$\pi_* \mathbb{F}^{MU} \otimes_{\mathbb{Z}} \mathbb{C} \cong SH^{-*}_{cw}(X; \mathbb{L}) \otimes_{\mathbb{C}} \mathbb{C}[b_1, \ldots].$$

Rmk.

In fact, we show more. There is a change-of-coefficients spectral sequence associated to $\mathbb{C}[\underline{b}] \to \mathbb{C}$, & non-degeneracy obstructs $\mathbb{F}^{MU} \cong \mathfrak{X} \otimes_{\mathbb{S}} MU$. We show the spectral sequence of the LHS is isomorphic to that of the RHS from E_2-page onwards. Theorem I follows by looking at the first non-trivial differential on the RHS.

Rmk

The relation to GRR | Atiyah-Singer is as follows. The linearization of Floer's equation is a family of Fredholm operators on the universal curve over the moduli. In particular, the LHS uses the Chern character of the index bundle & the RHS the Chern character of the $\mathbb{C}$-vector bundle the family is defined on.

Rmk

Theorem II is our "∞-categorical GRR". In particular, we do not use analysis to prove the coherence of GRR, we instead utilize the full ∞-categorical machinery of Abouzaid-Blumberg to upgrade everything to ∞-categories & ∞-functors:

$$\begin{tikzcd}
\mathrm{Flow}^{GRR} \arrow[r, "\mathrm{incl}"] \arrow[dr, bend right, "\text{bulk-deformations} = \text{“}\int_{\mathcal{U}|\mathbb{F}} ch(E)\text{”}"'] & \mathrm{Flow}^{\mathfrak{X}} \arrow[d, "\text{“}Ch\text{”} = \mathbb{C}\text{-linearization.}"] \\
& N^{dg}(\mathrm{Mod}_{\mathbb{C}[\underline{b}]})
\end{tikzcd} \quad (*)$$

Rmk

We can show $(*)$ commutes assuming the following symmetric monoidal structure on Flow.

§ 5. Various applications

Let us return to $T^*\mathbb{C}P^n$, $n \geq 3$ & odd.

Observe, this admits a standard $\mathbb{R}$-polarization:

$$T(T^*\mathbb{C}P^n) \cong \pi^* T\mathbb{C}P^n \otimes_{\mathbb{R}} \mathbb{C} \cong \Lambda_{std} \otimes_{\mathbb{R}} \mathbb{C}.$$

We have already noted

$$\mathbb{F}^{MU} \not\cong \mathbb{F}^{\mathbb{S},\Lambda_{std}} \otimes_{\mathbb{S}} MU.$$

Spectral Viterbo: Q a closed smooth manifold & $X = T^*Q$.

Thm^n | ($\mathbb{S}$-Viterbo)

$$\mathbb{F}^{\mathbb{S},\Lambda_{std}} \simeq \Sigma^{\infty}_{+} \mathcal{L}Q \in Mod_{\mathbb{S}}.$$

So, is there an MU-Viterbo?

Thm^n | (B.)

$\exists$ MU-local system $\Sigma^{-\dim Q}\nu \otimes \mathcal{E} : \mathcal{L}Q \to BGL_1 MU$ s.t.

$$\mathbb{F}^{MU} \simeq (\mathcal{L}Q)^{\Sigma^{-\dim Q}\nu \otimes \mathcal{E}} \in Mod_{MU}.$$

Rmk

The MU-local system is constructed via index theory. There is a conjectural homotopical incarnation. The case $X = T^*\mathbb{C}P^n$ shows it does not follow that $\Sigma^{-\dim Q}\nu \otimes \mathcal{E}$ lifts to $BGL_1\mathbb{S}$ all the time.

Spectral relative Gromov-Witten: $X = T^*\mathbb{C}P^n$ is the complement of a $(1,1)$ hypersurface D in $M = \mathbb{C}P^n_x \times \mathbb{C}P^n_y$: $D = \{\sum x_i y_i = 0\}$.

Let $S_D M$ be the circle bundle of the normal bundle of D in M.

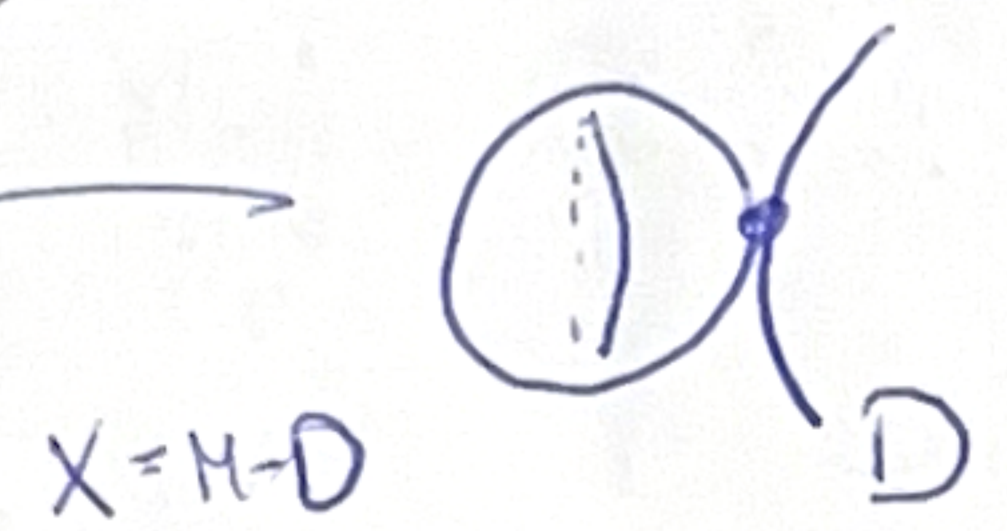
In particular, $\mathbb{F}^{MU}$ is a filtered spectrum w/ associated graded

$$\mathrm{Gr}_k\, \mathbb{F}^{MU} \simeq \begin{cases} \mathbb{D}\mathbb{CP}^n_+ \otimes_{\mathbb{S}} MU, & k=1 \\ \mathbb{D}\Sigma^{-2nk}(\mathbb{S}M)_+ \otimes_{\mathbb{S}} MU, & k>1 \\ *, & \text{otherwise,} \end{cases}$$

w/ $\mathbb{D}(-)$ denoting Spanier-Whitehead dual. The obstruction to $\mathbb{F}^{MU}$ splitting is given by

$$GW^{MU} \in \Omega_0^{Cx}\left(\Sigma^{-2n+1}(\mathbb{S}M)_+ \otimes_{\mathbb{S}} \mathbb{D}\mathbb{CP}^n_+\right)$$

which is related to complex cobordism classes of higher-dim 4-pointed genus 0 relative GW moduli ◁—



Rmk

The $\mathbb{Z}$-level statement of this is the content of Ganatra-Pomerleano '20.

by non-base-change computation above

essentially uses Bökstedt-Ottosen '18

It follows that $GW^{MU} \neq 0$, but $GW^{\mathbb{Z}} \equiv GW^{MU} \otimes_{MU} \mathbb{Z} = 0$. Hence, we have detected a spectral relative GW invariant.

Spectral mirror symmetry: The spectral open-closed map establishes an equivalence

here, take THH rel. MU $\left\{ THH(\mathrm{Fuk}(X; MU)) \simeq \mathbb{F}^{MU}, \right.$

under appropriate hypotheses. In particular, we should be able to find examples of the following (?)

(1) The mirror $\check{X}$ lives over MU & cannot be lifted to $\mathbb{S}$

(2) The mirror $\check{X}$ lives over MU & there is a mirror $\check{X}'$ over $\mathbb{S}$, but $\check{X} \neq \check{X}' \otimes_{\mathbb{S}} MU$.

Upshot: MU-Floer homotopy & $\mathbb{S}$-Floer homotopy are orthogonal in some sense.