

Complex cobordism & Grothendieck-Riemann-Roch in homotopical symplectic geometry

(1)

(based on joint work w/ Noah Powell)

§ 4. Reflections

Let $f: X \rightarrow Y$ be proper map of $\mathbb{C}$ -schemes. Recall, given $E \rightarrow X$ $\mathbb{C}$ -vector bundle,

Chern Character $\rightarrow$ $ch(E) \equiv e^{x_1} + \dots + e^{x_{rk} E}$, x_i i th Chern root of E .

Q:

Does the diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{f!} & K(Y) \\ \downarrow ch & & \downarrow ch \\ H^*(X; \mathbb{Q}) & \xrightarrow{f!} & H^*(Y; \mathbb{Q}) \end{array}$$

Commute?

A:

No!

Remark / (Grothendieck-Riemann-Roch)

$$ch(f!E) = f!(ch(E) \cup Td(TX/Y)).$$

Remark

$Td(TX/Y)$ is a "correction term":

$$Td(E) \equiv \prod_{i=0}^{rk E} Q(x_i), \quad Q(t) \equiv \sum_{m=0}^{+\infty} \frac{B_m}{m!} t^m \text{ w/ } B_m \text{ mth Bernoulli number.}$$

The family Atiyah-Singer theorem essentially asks:

"What is the differential-geometric analogue of GRR?"

Let $\Theta \equiv \mathbb{R}_3 \times S^4_t$; we will restrict to $\pi: U \equiv X \times \Theta \rightarrow X$, where X is oriented, closed smooth manifold. Consider:

- $E \rightarrow U$ $\mathbb{C}$ -vector bundle w/ fixed trivializations over tube-like ends,

- $e_{\pm} \in \Omega^{0,1}(\Theta, \text{End}_{\mathbb{C}}(\text{core } E))$ s.t. translation-invariant $\dot{e}$

$$\bar{\partial} + e_{\pm}: P(\underline{\mathbb{C}}^{\text{core } E}) \rightarrow \Omega^{0,1}(\Theta, \underline{\mathbb{C}}^{\text{core } E})$$

is an isomorphism, $\dot{e}$

- $D \equiv \bar{\partial}_E + \gamma: P(E) \rightarrow \Omega^{0,1}(E)$ s.t.

i. $\bar{\partial}_E \equiv \bar{\partial}$, ∇ Hermitian connection on E ,

ii. $\gamma \in \Omega^{0,1}(E, \text{End}_{\mathbb{C}}(E))$ agrees w/ $e_{\pm}$ resp. e_{-} over $+$ resp. $-$ tube-like end.

Now, after appropriate Sobolev completion, D is a family of Fredholm operators

$$D = \{ \bar{\partial}_{E_p} + \gamma|_{E_p} \}_{p \in X}, \quad E_p \rightarrow \pi^{-1}(p) = \Theta,$$

parameterized by X , each of (Fredholm) index

$$\text{deg}(e_{-}) - \text{deg}(e_{+}), \quad \text{deg}(e_{\pm}) \equiv \text{rk}_{\mathbb{C}} E - \text{ch}(e_{\pm}).$$

In particular, $\text{Ker } D_x \oplus \text{coker } D_x$ finite-dimensional. In fact, we define $\text{ind } D \equiv \text{Ker } D - \text{coker } D \in \mathbb{Z}(X)$.

Rmk

Technically, this only works when $\text{Ker } D \oplus \text{coker } D$ constant rank... can do in general anyways.

Thm (family Atiyah-Singer index: special case)

$$\underbrace{\text{Ch}(\text{ind } D)}_{\text{"analytic index"}} = \underbrace{\pi_! \text{Ch}(E)}_{\text{"topological index"}}$$

$$* \pi_!(\text{Th} E) = \eta *$$

Rmk

Should think $\text{ind } D = \pi_! E$.

So... what's the problem?

§ 2. Fiber-theoretic considerations

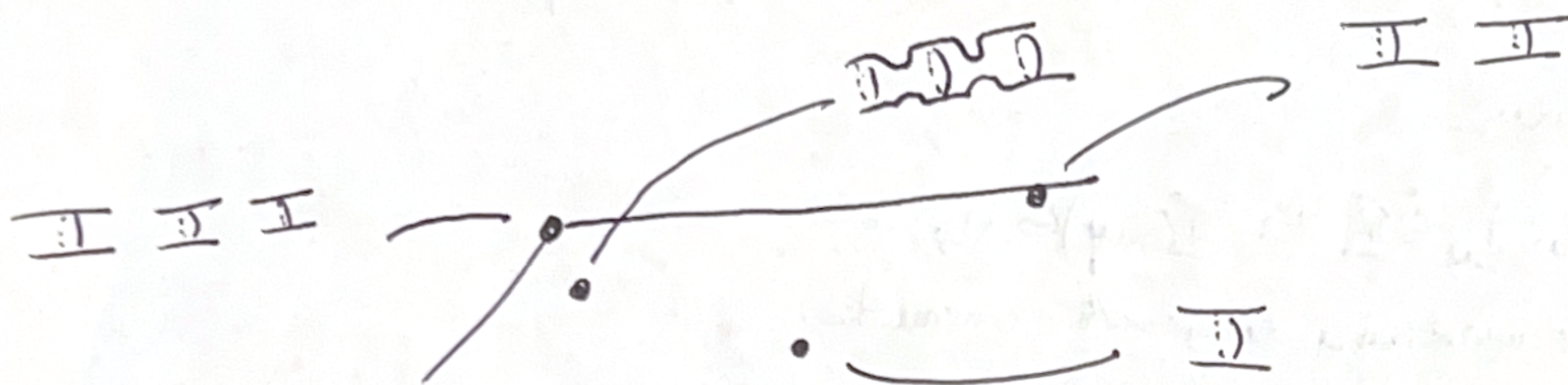
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So, when X oriented, closed, duality $A-D = D \int_X (\text{rep. of } \text{Ch}(\text{incl } D)) = \int_X (\text{rep. of } \pi_! \text{ch}(E))$.

But, in Fiber theory, we are interested in the scenarios where:

- i. X is oriented, compact smooth manifold w/ corners ε !
- ii. the fibers of $\mathcal{U}$ degenerate over the corner strata.

In this case, the integral will depend on representative due to boundary terms, & we have this degeneration issue.



Defn / (flow category)

A flow category consists of the following data.

- Objects: directed $\mathcal{D}$
- morphisms: compact smooth manifolds w/ corners $\overline{F}_{xy}$
- Composition: embeddings of codim 1 boundary strata $\overline{F}_{xz} \times \overline{F}_{zy} \hookrightarrow \overline{F}_{xy}$

moreover, codim 1 boundary strata of $\overline{F}_{xy}$ enumerated by such maps.

Require that all natural associativity diagrams commute.

Examples:

$\mathcal{U}$: single object $*$, no morphisms

$\mathcal{M}_{\text{fig}}$: (fig) Morse-Smale pair on closed smooth manifold Q .

Defn (complex-oriented flow category)

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A complex-oriented (cx) flow category is a flow category $\mathcal{F}$ w/ the following additional data.

- Real virtual vector spaces V_x .

- $\mathbb{C}$ -virtual bundles $I(xy) \rightarrow \mathbb{F}_{xy}$ s.t.

$$I(xy) \cong I(x,z) \oplus I(z,y) \text{ over } \mathbb{F}_{xz} \times \mathbb{F}_{zy}.$$

- Equivalences

$$\mathbb{F}_{xy} \oplus V_y \oplus \mathbb{R} \cong I(xy) \oplus V_x.$$

We require all natural associativity diagrams commute.

Example:

$\Sigma^d \mathcal{U}$: single object $*$, no morphisms, $V_* \cong \mathbb{R}^d$.

We have

(cx) flow categories = "(cx) flow 0-simplices".

Can easily define "(cx) flow n-simplices"; I won't do this in general.

A cx flow bimodule (flow 1-simplex)

$$W: \Sigma^d \mathcal{U} \rightarrow \mathcal{U}$$

is simply a closed smooth manifold W_{**} together w/

$$\overline{W}_{**} \cong I(*,*) + \mathbb{R}^d,$$

i.e., W_{**} is equipped w/ a stable complex structure! A homotopy

$$H: W^0 \rightarrow W^1, \quad W^0, W^1: \Sigma^d \mathcal{U} \rightarrow \mathcal{U}$$

is simply a cx cobordism $H_{**}: W^0 \rightarrow W^1$.

In particular, there is a stable co-algebra of CX flow categories, Flow^{CX} , which is equivalent to Mod_{MU} w/ MU the complex cobordism spectrum, cf. Adams-Bloomberg-Hedenlund-Oderick. In fact, we say

$$[\Sigma^d \mathbb{Q}_1 \mathbb{Q}_1]_{\text{Flow}^{CX}} \cong \pi_0 \text{Flow}^{CX}(\mathbb{Q}_1, \mathbb{Q}_1) \cong \pi_0 MU \cong \mathbb{Q}_2^X \cong \mathbb{Z} \Rightarrow \pi_* \text{Flow}^{CX}(\mathbb{Q}_1, \mathbb{Q}_1) \cong \mathbb{Z}[b_1, b_2, \dots] \quad |b_i| = 2i.$$

Rmk
Can define $\text{Flow}^{\text{any cobordism}}$... e.g. oriented, stg, framed, etc.

Okay, back to GRR.

§ 3. co-algebraic structure of GRR

In Fuchs theory, the tangent bundle of a moduli space is usually related to the index bundle of a family of Fredholm operators ... This led Porcelli & me to "GRR flow categories". Let $\mathcal{F}$ be a flow category.

data for
GRR flow
cat

Universal curves:
 $\pi_{xy}: U_{xy} \cong \text{int } \mathcal{F}_{xy} \times \Theta \rightarrow \text{int } \mathcal{F}_{xy}$ s.t.
 $U_{xy} \cong U_{xz} \#_{1/\mathbb{Z}} U_{zy}$ over $\text{int } \mathcal{F}_{xz} \times \text{int } \mathcal{F}_{zy} \times (q \in \mathbb{Z})_{\mathbb{Z}}$

$\mathbb{C}$ -vector bundles:
 $E_{xy} \rightarrow U_{xy}$ $\mathbb{C}$ -vector bundles w/ fixed trivializations over tube-like arcs & "universal line" $\text{int } \mathcal{F}_{xy} \times (\mathbb{R} \times \mathbb{Z})$ s.t.
 $E_{xy} \cong E_{xz} \#_{1/\mathbb{Z}} E_{zy}$

Puncture data:
 $e_x \in \Omega^{q+1}(\Theta; \text{End}_{\mathbb{C}}(\mathbb{C}^{rk E_{xy}}))$ s.t. Hodge index invariant & $\bar{\partial}$ -ex universal form.

Fredholm operators:
 $D_{xy} \equiv \bar{\partial} E_{xy} + \gamma_{xy}$ s.t. γ_{xy} agrees w/ e_x resp. e_y over -resp. + tube-like arc!
 $D_{xy} \cong D_{xz} \#_{1/\mathbb{Z}} D_{zy}$

Conditions:
 (1) $\pi^* \mathcal{F}_{xy} + \mathbb{R} \cong D_{xy}$.
 (2) All natural associativity diagrams commute.

Atiyah: Chern-Weil Theory

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Given $(E, \nabla) \rightarrow X$, we may rep. $Ch(E)$ via

$$Ch(E; \nabla) \equiv \text{Tr} \left(\exp \left(\frac{-1}{2\pi i} F \nabla \right) \right) \in \Omega^*(X; \mathbb{C}),$$

where $F \nabla \in \Omega^2(X; \text{End}(E))$ is the curvature of ∇ . let $\mathbb{C}[b] \equiv \mathbb{T}^{\text{MU}} \otimes_{\mathbb{Z}} \mathbb{C}$. Consider

$$Ch(E; \mathbb{C}[b]) \equiv \sum_{p \geq 0} Ch_p(E; \nabla) b_p \in \Omega^0(X; \mathbb{C}[b]) \text{ w/ } Ch_p \text{ the graded piece of } Ch,$$

$$\exp(Ch(E; \mathbb{C}[b])) \equiv \sum_{n \geq 0} \frac{1}{n!} Ch(E; \mathbb{C}[b])^n.$$

Again, we have a stable ∞ -category Flow^{GRR} . There are three natural ∞ -functors.

1. $\text{Flow}^{\text{GRR}} \rightarrow \text{Flow}^{\text{CX}}, \quad \mathbb{F} \mapsto \text{"ind } \mathbb{F}"$

i. $\text{ob ind } \mathbb{F} = \text{ob } \mathbb{F}$

iii. $V_x \equiv \mathbb{R}\text{-deg}(e_x)$

ii. $(\text{ind } \mathbb{F})_{xy} = \mathbb{F}_{xy}$

iv. $I(xy) \equiv \text{ind } D_{xy}$

technically, need to stabilize

2. $\text{Flow}^{\text{CX}} \rightarrow \mathcal{N}^{\text{Ch}}(\text{Mod } \mathbb{C}[b]), \quad \mathbb{F} \mapsto \text{"Ch } \mathbb{F}"$

i. Choose coherent Hermitian connections ∇_{xy} on $I(xy)$ (i.e., $\nabla_{xy} = \nabla_{xz} \#_{V_C} \nabla_{zy}$)

we see $m_{xy}^I \equiv \exp(Ch(I(xy); \mathbb{C}[b]))$ satisfies

$$m_{xy}^I \Big|_{\mathbb{F}_{xz} \times \mathbb{F}_{zy}} = m_{xz}^I \wedge m_{zy}^I.$$

ii. Define Chern complex

$$Ch \mathbb{F} \equiv CM_*(Ch \mathbb{F}; m^I) \equiv \bigoplus_{x \in \mathcal{P}} \mathbb{C}[b] \langle x \rangle$$

w/ differential

$$\partial x \equiv \sum_{y \in \mathcal{P}} \left(\int_{\mathbb{F}_{xy}} m_{xy} \right) y.$$

3. $\text{Flow}^{\text{GRR}} \rightarrow \mathcal{N}^{\text{Ch}}(\text{Mod } \mathbb{C}[b]), \quad \mathbb{F} \mapsto \text{"} \int_{\mathcal{M}/\mathbb{F}} Ch(E) \text{"}$

Choose coherent Hermitian connections ∇_{xy} on E_{xy} & repeat 2. using

$$m_{xy}^E \equiv \exp(\pi_{xy, !} Ch(E_{xy}; \mathbb{C}[b])).$$

instead.

Thmn | (B.-Porcelli) (Friedholm model for MU)

$\text{Flow}^{\text{GRR}} \rightarrow \text{Flow}^{\text{G}}$ is an equivalence

Pr.
Essentially Bott periodicity à la Atiyah.

Rmk

Hence, $\text{Flow}^{\text{GRR}}(\mathbb{Q}, \mathbb{Q}) \cong \text{MU}$ is a Friedholm operator model for MU!

Thmn | (B.-Porcelli) (∞ -categorical GRR)

$$H_* \left(\int_{\text{MU}/\mathbb{F}} \text{Ch}(\mathbb{E}) \right) \cong H_* \left(\text{Ch}(\text{incl}(\mathbb{F})) \right).$$

Pr.

The diagram of ∞ -functors

$$\begin{array}{ccc} \text{Flow}^{\text{GRR}} & \longrightarrow & \text{Flow}^{\text{G}} \\ & \searrow & \downarrow \\ & & \text{Ndg}(\text{Mod}(\mathbb{Q}[k])) \end{array} \quad (*)$$

is a diagram of equivalences after tensoring w/ $\mathbb{Q}$. Hence, commutes up to an autoequivalence of $\text{Ndg}(\text{Mod}(\mathbb{Q}[k]))$... Then classified such autoequivalences as given by tensoring w/ an invertible bimodule. We classify the bimodule & use the usual GRR.

Rmk:

I.e., we do not show the integrals $\int_{\mathbb{F}_{xy}} m^{\mathbb{E}}_{xy} \quad \& \quad \int_{\mathbb{F}_{xy}} m^{\mathbb{I}}_{xy}$ are related via analogies... our proof is pure homotopy theory. One could try to do the analysis e.g. showing the heat kernel proof is coherent. We thought this was too hard.

"Conjecture (B.-Porcelli)"

The diagram (*) commutes.

Well, its "geometric" construction...
i.e., not just transfer from MU to MU

Rmk

We have a proof contingent on the (conjectural) symmetric monoidal structure on Flow^{GRR} resp. Flow^{G} . But, it is essentially known how to construct this, hence the question marks.