

Introduction to Lagrangian Floer theory, II: the closed case

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In last week's pre-talk, I introduced Lagrangian Floer cohomology in a simpler setting: the exact case. In this talk, we'll talk about how to deal w/ issues the closed case brings.

References: Fucc, Auroux's "A Beginner's intro..." & 2020 ICM.

§ 1. Novikov: the compactness issue, etc.

(M, ω) closed symplectic manifold

Consider Riemann surface Σ ($\partial\Sigma \neq \emptyset$) w/ 2 punctures, a 1-form $\alpha \equiv X_H \otimes \beta$ on Σ valued in Hamiltonian vector fields on M (which is $\mathbb{R}t, X_H \otimes dt$, over strip-like ends), & a (possibly Σ -dependent) ω -compatible a.c.s J on M .

We may look at moduli space of soln's to Floer's eqn,

$$(du - \alpha)^{0,1} = 0,$$

of finite energy,

$$E(u) \equiv \int_{\Sigma} u^* \omega - u^* dH \wedge \beta,$$

w/ asymptotic & Lagrangian 2 conditions.

In exact setting, we had an a priori $E(u)$ bound via the action — here, we don't necessarily have this. Let K be a field (e.g. $\mathbb{Z}_2, \mathbb{Q}, \mathbb{R}, \mathbb{C}$)

need assumptions to orient moduli.

The Novikov ring is

$$\Lambda_0 \equiv \left\{ \sum_{i=0}^{+\infty} a_i T^{\lambda_i} : a_i \in K, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i \rightarrow +\infty} \lambda_i = +\infty \right\}.$$

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This has a non-Archimedean valuation:

$$v(0) \equiv +\infty$$

$$v\left(\sum a_i T^{\lambda_i}\right) \equiv \min \{ \lambda_i : a_i \neq 0 \}.$$

Essentially, Lagrangian Floer cohomology will be a Λ -module...

using Λ allows us to deal w/ $E(u)$.

§ 2. Virtual techniques: the transversality issue.

We won't spend too much time on this.

$L_0, \dots, L_d \subset M$ closed Lagrangians.

(2)

Assume $L_i \cap L_{i+1}$ (cyclic) for simplicity... if not, use Hamiltonians
Goursin

$$p_i \in L_i \cap L_{i+1}, 1 \leq i \leq d, \quad q \in L_d \cap L_0.$$

We consider the moduli space

$$\mathcal{M}(p_1, \dots, p_d, q, \beta, \bar{J})$$

of soln's

$$u: \Sigma_d \rightarrow M, \quad \bar{\partial}_{\bar{J}} u = 0 \quad \& \quad u_*[\Sigma_d] = \beta,$$

where:

- Σ_d is unit disk in $\mathbb{C}$ w/ $d+1$ marked
pts $z_0, \dots, z_d \in \partial \Sigma$ oriented counterclockwise;
- $\bar{J}$ as before;
- $u(z_i) = p_i \quad \& \quad u(z_0) = q$;
- $u(\partial \text{ arc connecting } z_i \text{ to } z_{i+1}) \subset L_i$.

This is a Fredholm problem, i.e., $\bar{\partial}$ bundle $E \rightarrow \mathcal{F}$ over Banach
manifold w/ section $\bar{\partial}_{\bar{J}}: \mathcal{F} \rightarrow E$ s.t.

$$\bar{\partial}_{\bar{J}}^{-1}(0) = \mathcal{M}(p_1, \dots, p_d, q, \beta, \bar{J}).$$

If like was kind, the Fredholm operator $\bar{\partial}_{\bar{J}}$, for generic $\bar{J}$, would be surjective -
hence, moduli would be transversely cutout of expected dimension and $\bar{\partial}_{\bar{J}}$.

But life sucks :(Fortunately, there are techniques (Kuranishi stuff)
which allow us to extract a Virtual fundamental class (VFC) (aka competing),
lying in the expected degree, in order to "count" (in $\mathbb{Z}_2$ generally,
over $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ if e.g. oriented spin).

In any event, we may define

$$CF(L_i \cap L_{i+1}) \equiv \bigoplus_{x \in L_i \cap L_{i+1}} \langle x \rangle$$

$\&$

$$\begin{aligned} \mu_d: CF(L_{d+1}, L_d) \otimes \dots \otimes CF(L_0, L_1) &\rightarrow CF(L_0, L_d) \\ (p_d, \dots, p_0) &\mapsto \sum_{\substack{q \in L_d \cap L_0 \\ \text{ind } \beta = 2}} |\mathcal{M}(p_1, \dots, p_d, q, \beta, \bar{J})| \tau(E(\beta))_q. \end{aligned}$$

* Gromov compactness says that, $\forall E_0$, $\exists$ only finitely many β s.t. $E(\beta) \leq E_0 \quad \& \quad \mathcal{M}(p_1, \dots, p_d, q, \beta, \bar{J}) \neq \emptyset$, so the RHS is well-defined.

Consider $\mu_1: CF(L_0, L_1) \rightarrow CF(L_0, L_1)$.

Q: Is $\mu_1^2 = 0$?

§ 3. Curvature: the compactness issue, pt ii

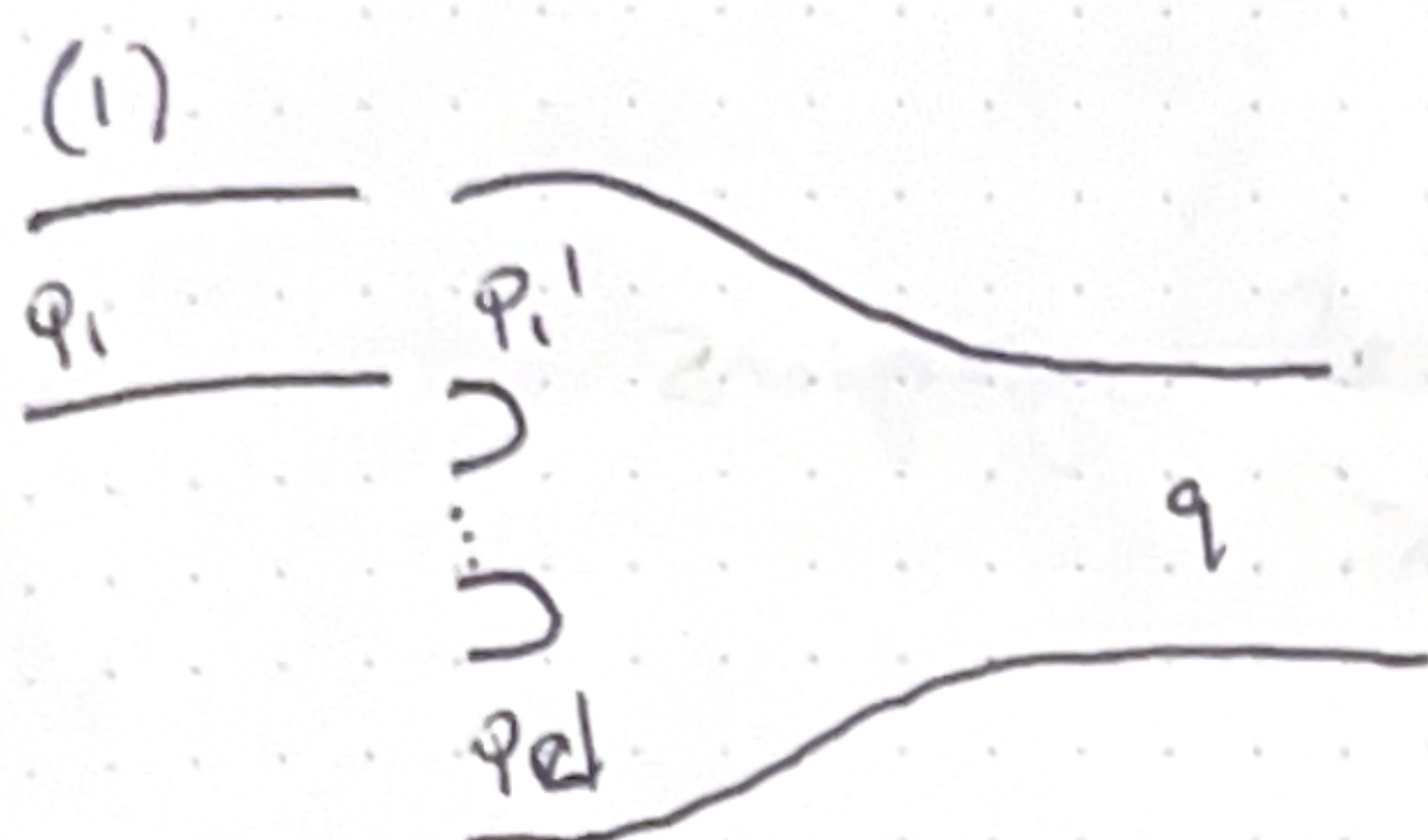
(3)

When compactifying $M(p_1, p_2, q, \mathbb{R}^+)$, three things can happen.

(1) Strip breaking: energy concentrates at infinity in a strip like end.
— This is what we want.

(2) Sphere bubbling: energy concentrates at an interior pt.
— This is codimension 2, we'll live.

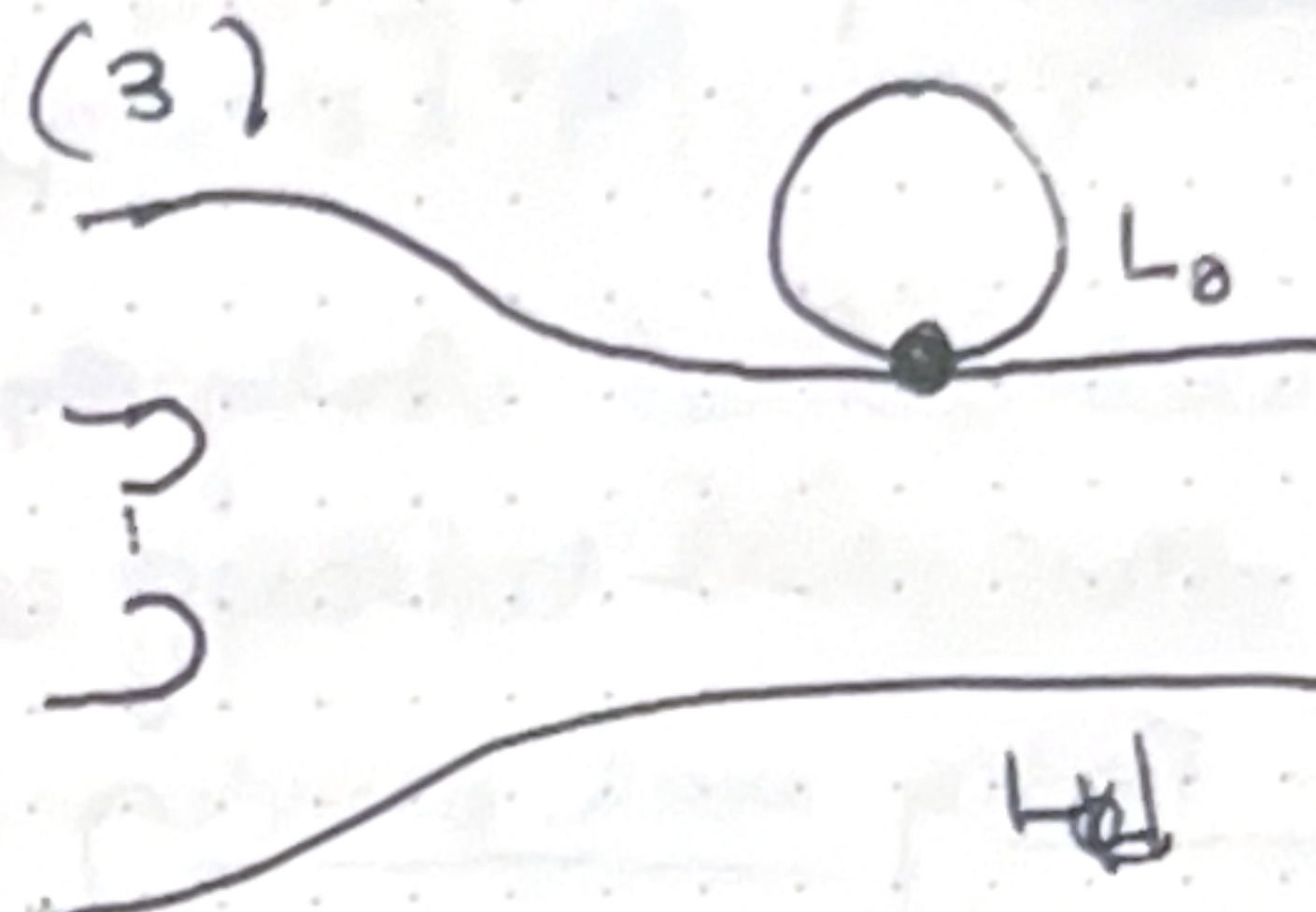
(3) Disk bubbling: energy concentrates at ~~an~~ 2 pt
— This is codimension 1, oh no.



Strip breaking



Sphere bubbling



Disk bubbling

⚠ (3) is a genuine feature which obstructs $\mu_1^2 = 0$.

So, where are we at?

In general,

$$\mu_1^2(p) \pm \mu_2(\mu_0^{L_1}, p) \pm \mu_2(p, \mu_0^{L_0}) = 0,$$

where μ_0^L is a count of disks w/ ∂ on L .

These are a special case of the curved A_∞ -relations:

$$\sum_{\substack{0 \leq k \leq d \\ 0 \leq j \leq d-k}} (-1)^* \mu_{d+1-k}(p_{d+1}, p_{j+k+1}, \mu_k(p_{j+k}, \dots, p_{j+1}), p_j, \dots, p_1) = 0,$$

w/ * Some sign.

When $\mu_0 = 0$, these are the (ordinary) A_∞ -relations. ($\mu_0^L = 0$ when e.g. $\omega|_{\pi_2(M, L)} = 0$).

Observe, we've essentially written down the Fukaya category.

§ 4. Dealing w/ curvature: (weak) bounding cochains

Q. Is there a way to "get rid of curvature"? I.e., can we define new operations s.t. they satisfy the (ordinary) A_∞ -relations.

This is the concept of a (weak) bounding cochain introduced by Fukaya-Ono-Ohta-Ono.

An element $b \in CF(L, L)$ (odd degree, positive Novikov valuation) is a bounding cochain for L if

$$\mu_0^b \equiv \sum_{k \geq 0} \mu_k^b(b^{\otimes k}) = 0 \in CF(L, L).$$

(4)

We say L is unobstructed if it admits a bounding cochain.

Analogously, $b \in CF(L, L)$ is a weak bounding cochain if

$$\mu_b^0 = \lambda \mathbb{1}_L \in CF(L, L), \quad \lambda \in \mathbb{K}.$$

We say L is weakly unobstructed if it admits a weak bounding cochain.

Given $L_0, \dots, L_d$ as before w/ bounding cochains b_i , we define

$$\mu_d^b(p_d, \dots, p_1) = \sum_{k_0, \dots, k_d \geq 0} \mu_{d+k_0+\dots+k_d}^{\text{bulk}}(b_d^{\otimes k_d}, p_d, \dots, b_1^{\otimes k_1}, p_1, b_0^{\otimes k_0});$$

these satisfy the (ordinary) A_∞ -relations!

In particular,

$$HF((L_0, b_0), (L_1, b_1))$$

is well-defined. Analogously, we can use weakly unobstructed Lagrangians if their weak bounding cochains have the same $\lambda \in \mathbb{K}$.

§ 5. Finding weak bounding cochains: bulk deformations

Essentially, ~~it is easier to find a~~ weak bounding cochain for L after "bulk deformation".

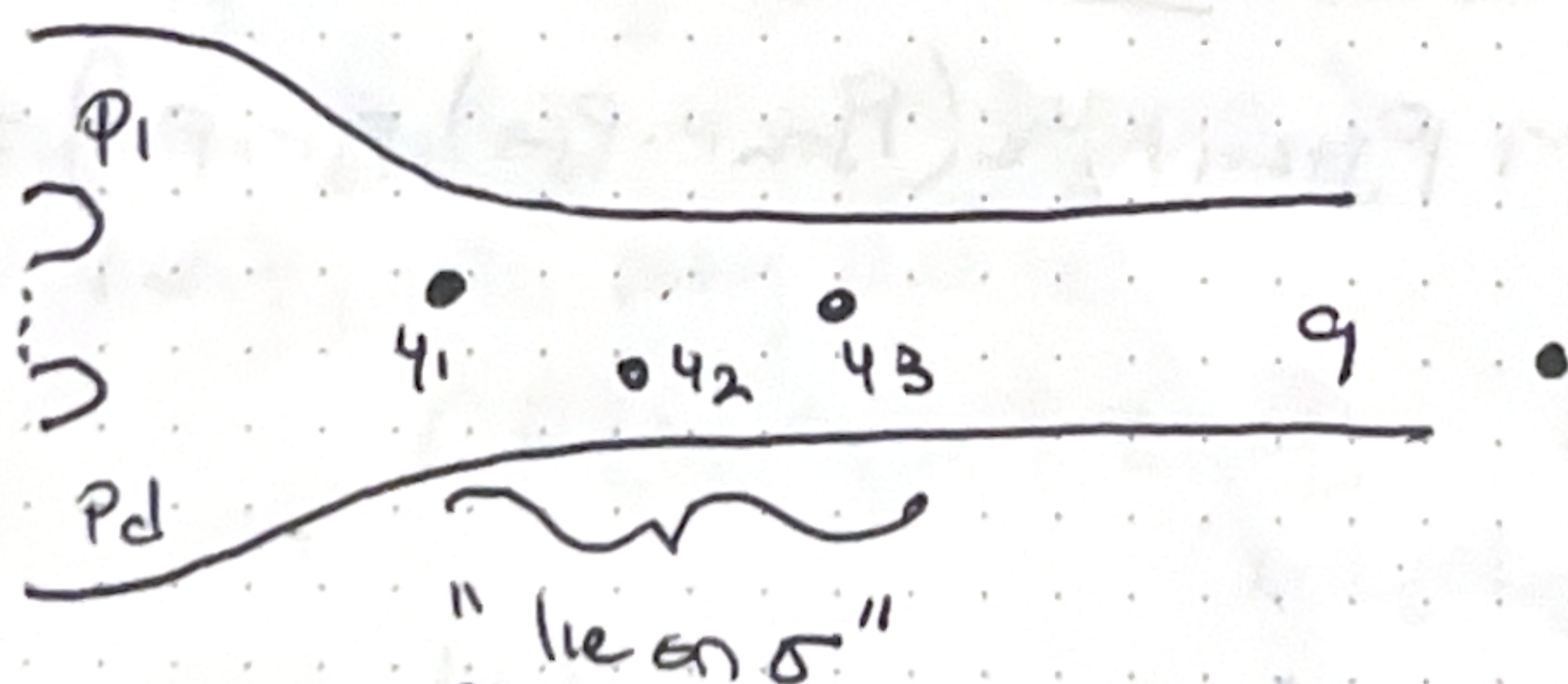
Let $\sigma \in H^2(M; \mathbb{K})$. Consider $\Sigma_{d, \ell}$: this is the unit disk in $\mathbb{C}$ w/

- $d+1$ marked pts $z_0, \dots, z_d \in \partial \Sigma_{d, \ell}$, oriented counterclockwise;
- ℓ interior marked points $y_1, \dots, y_\ell \in \text{int } \Sigma_{d, \ell}$.

We now define

$$\mu_d^\sigma \equiv \mu_d + (\text{1st term}) + (\text{2nd term}) + \dots,$$

where the ℓ^{th} term counts disks like μ_d but the y_j 's are required to pass through ~~the~~ smooth representative of the Poincaré class of σ .



Now, given L , ~~can look for~~ bulk-deformation $\{\mu_d^\sigma\}$ s.t. L is (weakly) unobstructed w.r.t. this bulk-deformation.

§ 6. Extra: local systems

In homological mirror symmetry, one is naturally led to consider Lagrangians equipped w/ local systems, i.e., we take $E \rightarrow L$ a flat vector bundle w/ unitary holonomy over Λ w/ $\mathbb{K} = \mathbb{C}$. In particular, holonomy is of the form

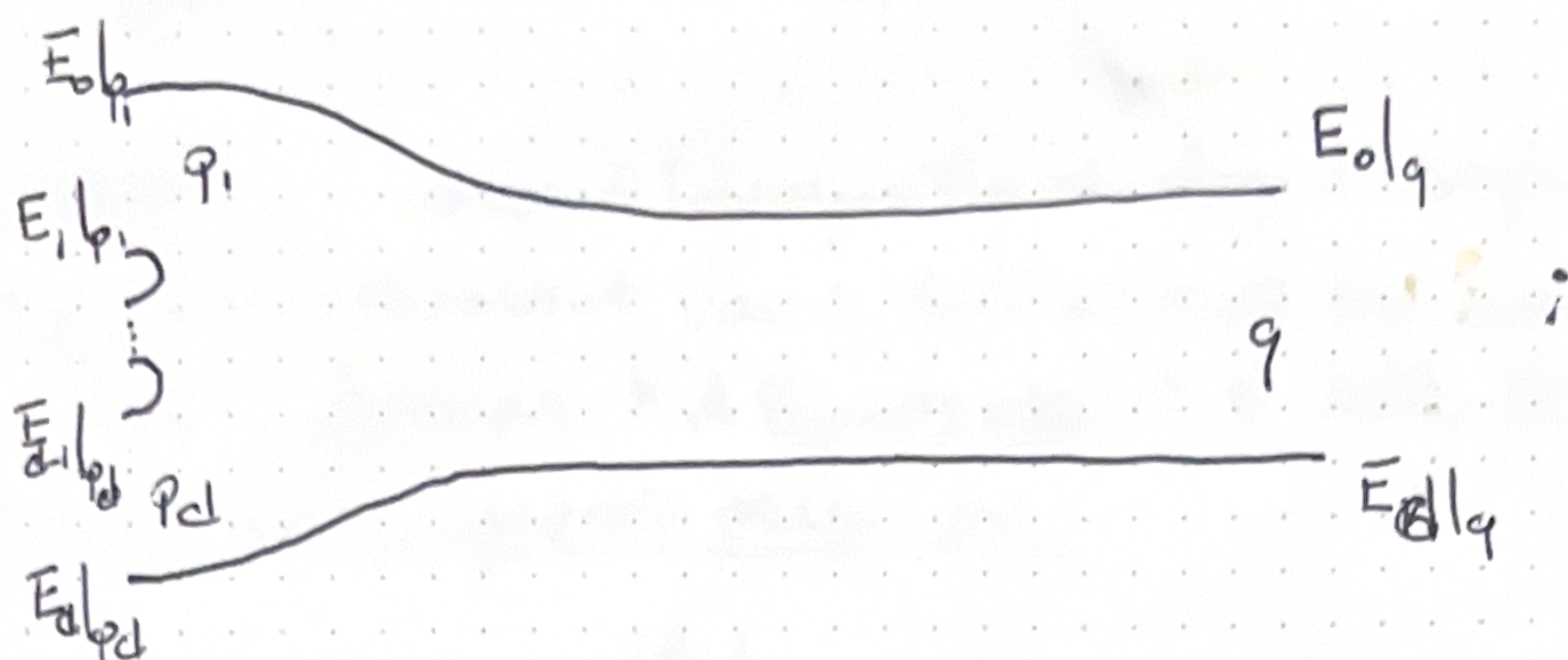
$$a_0 + \sum a_i T^{h_i}, \quad a_0 \text{ invertible}, \quad h_i > 0 \text{ } \forall i.$$

We define

(5)

$$CF((L_i, E_i), (L_{i+1}, E_{i+1})) \equiv \bigoplus_{x \in L_i \cap L_{i+1}} \text{hom}(E_i|_x, E_{i+1}|_x)$$

Consider



by parallel transporting around disc U as above ($\omega|_{U_x}[\Sigma_d] = \beta$), we get an element $\eta_{\beta, p, p_d, q}$. We define

$$\mu_d(\eta_d, \dots, \eta_1) \equiv \sum_{\substack{q \in L_1 \cap L_0 \\ \text{ndp} = 2-d}} |\mathcal{M}(p_1, \dots, p_d, q, \beta, \mathcal{J})| T^{E(\beta)} \eta_{\beta, p_1, \dots, p_d, q}$$

We can also work deform in this context.